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On the Leading Coefficients of Real Many-Variable Polynomials

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For a homogeneous polynomial P in N variables, $x_1, ..., x_N$, of degree k, the leading terms are those which contain only one variable, raised to the power k. If $0 \leqslant P \leqslant 1$ when all variables satisfy $0 \leqslant x_j \leqslant 1$, how large can the leading coefficients be? Estimates were given by R. Aron, B. Beauzamy, and P. Enflo (*J. Approx. Theory* **74** (2) (1993), 181–198); we improve these estimates in general and solve the problem completely for $k = 2$ and 3. Symbolic computation (MAPLE on a *Digital* DecStation 5000) was heavily used at two levels: first in order to get a preliminary intuition on the concepts discussed here, and second, as symbolic manipulation on polynomials, in most proofs. Numerical analysis was made on a Connection Machine CM2, using the hypercube representation obtained by B. Beauzamy, J.-L. Frot, and C. Millour (Massively parallel computations on many-variable polynomials: When seconds count, preprint).

Let

$$P(x_1, ..., x_N) = \sum_{|\alpha| = k} a_\alpha x_1^{\alpha_1} \cdots x_N^{\alpha_N}, \tag{1}$$

with $\alpha = (\alpha_1, ..., \alpha_N)$, $|\alpha| = \alpha_1 + \cdots + \alpha_N$, be a homogeneous polynomial of degree k in N variables $x_1, ..., x_N$.

As already done by Aron, Beauzamy, and Enflo in [1], among all coefficients a_α, we distinguish the *leading* ones, denoted by a_l ($l = 1, ..., N$): a_l is the coefficient of the sole variable x_l, raised to the power k. So the *leading terms* are those which contain just one variable, raised to the power k (this terminology is of course inspired by the one-variable situation). All other terms contain at least two variables, and the polynomial can be written

$$P(x_1, ..., x_N) = \sum_{l=1}^{N} a_l x_l^k + \sum_{|\beta| = k} a_\beta x_1^{\beta_1} \cdots x_N^{\beta_N}, \tag{2}$$

where in the last term all β's have at least two non-zero components.

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The question raised in [1] is: If we know $\sum_{l=1}^{N} |a_l|$, can we find a lower bound for $\max_{0 \leqslant x_j \leqslant 1} |P(x_1, ..., x_N)|$?

The reason for this question, explained in [1], is that such a result allows us to decrease the number of terms in the polynomial we need to consider: in order to find a lower bound for $\max_{0 \leqslant x_j \leqslant 1} |P(x_1, ..., x_N)|$ (a quantity which depends on *all terms* in P), we need only to consider the a_l's, which represent only N terms.

In [1] it was shown that

$$\sum_{1}^{N} |a_l| \leqslant C_k \max_{0 \leqslant x_j \leqslant 1} |P(x_1, ..., x_N)| \tag{3}$$

with $C_k \leqslant 4k^2$, and that the best C_k must satisfy $C_k \geqslant k$.

We will obtain estimates for C_k from below by an iterative procedure (at each step, replacing the variable by a previously obtained polynomial), and this iterative procedure will require $0 \leqslant P \leqslant 1$ when all variables satisfy $0 \leqslant x_j \leqslant 1$ (and not just $|P| \leqslant 1$). For this technical reason, we investigate

$$B_k = \sup \left\{ \sum_{l=1}^{N} |a_l|;\ P \text{ as in } (1),\ 0 \leqslant P \leqslant 1 \text{ if } \quad 0 \leqslant x_j \leqslant 1, \right.$$
$$\left. j = 1, ..., N;\ N = 1, 2, ... \right\} \tag{4}$$

and we want to find a lower bound for B_k. We write $|P|_{\text{lead}}$ instead of $\sum_{l=1}^{N} |a_l|$.

Our main result is:

THEOREM 1. *For $k \geqslant 2$, the following estimates hold: $B_2 \geqslant 4$, $B_3 \geqslant 9$, and for $k \geqslant 3$,*

$$B_k \geqslant k^{\log 6/\log 3}.$$

Each of these estimates requires the production of a corresponding polynomial. The techniques are different in each case.

PROPOSITION 2. *The polynomial in N variables, homogeneous of degree 2,*

$$P(x_1, ..., x_N) = A \left(\frac{x_1^2 + \cdots + x_N^2}{N} - \frac{2}{N(N-1)} \sum_{i<j} x_i x_j \right),$$

with $A = 4(N-1)/N$ if N is even, $A = 4N/(N+1)$ if N is odd, satisfies $0 \leqslant P \leqslant 1$ if all x_j's satisfy $0 \leqslant x_j \leqslant 1$.

Proof of Proposition 2. Let's first show that $P \geqslant 0$, that is,

$$\frac{2}{N(N-1)} \sum_{i<j} x_i x_j \leqslant \frac{1}{N} (x_1^2 + \cdots + x_N^2)$$

or

$$\frac{1}{N(N-1)} \left(\sum_{i \neq j} x_i x_j + \sum_i x_i^2 \right) \leqslant \left(\frac{1}{N} + \frac{1}{N(N-1)} \right) \sum_i x_i^2,$$

which is equivalent to

$$\frac{1}{N} \left(\sum_i x_i \right)^2 \leqslant \sum_i x_i^2,$$

a consequence of Hölder's inequality.

Let's now show that $P \leqslant 1$. Since P is a convex function of each x_j, it's enough to show it when $x_j = 0$ or 1.

Let's assume that K of the x_j's are 1, and $N-K$ are 0.

The condition $P \leqslant 1$ reads

$$A \left(\frac{K}{N} - \frac{2}{N(N-1)} \frac{K(K-1)}{2} \right) \leqslant 1,$$

or

$$A \frac{K(N-K)}{N(N-1)} \leqslant 1.$$

The maximum of $K(N-K)/N(N-1)$ is reached for $K \sim N/2$. More precisely, if N is even, $N = 2M$, it is obtained for $K = M$, and gives $N/4(N-1)$, and if $N = 2M+1$, it is obtained for $K = M$, and gives $(N+1)/4N$. The values of A follow.

This gives the required estimate for B_2 in Theorem 1. We observe that the maximum is not obtained for a fixed number of variables, but letting $N \to +\infty$.

We now turn to the case of degree 3:

PROPOSITION 3. *The polynomial*

$$P(x_1, ..., x_N) = A \sum_1^N x_i^3 - B \sum_{i \neq j} x_i^2 x_j + C \sum_{i<j<k} x_i x_j x_k, \tag{5}$$

with

$$A = \frac{1}{N^3}(3N-4)^2,$$

$$B = \frac{8}{N^3}(3N-6),$$

$$C = \frac{96}{N^3},$$

satisfies $0 \leqslant P \leqslant 1$ *when all the* x_j*'s satisfy* $0 \leqslant x_j \leqslant 1$.

Assuming this result, we see that the estimate for B_3 follows: indeed, when $N \to +\infty$, $|P|_{\text{lead}} \to 9$. But here again, the maximum is not reached for any prescribed number of variables.

Proof of Proposition 3. We take P under the form (5) and compute the values of A, B, C, so it has the required properties. First, we study the case where K of the variables x_j take the value 1, and $N-K$ take the value 0 $(0 \leqslant K \leqslant N)$. We set

$$\varphi(K) = P(\underbrace{1, ..., 1}_{K \text{ times}}, 0, ..., 0)$$

and so

$$\varphi(K) = AK - BK(K-1) + \frac{C}{6} K(K-1)(K-2). \tag{6}$$

We will choose A, B, C, such that $0 \leqslant \varphi(K) \leqslant 1$ for $K = 0, ..., N$, and with A as large as possible since $|P|_{\text{lead}} = AN$.

The polynomial $\varphi(x)$ must vanish at 0, must satisfy $0 \leqslant \varphi(x) \leqslant 1$ if $x \in [0, N]$, and we want $\varphi(1)$ to be as large as possible. Therefore, we will require φ to have a double zero α, $0 \leqslant \alpha \leqslant N$ (which ensures $\varphi(x) \geqslant 0$, $0 \leqslant x \leqslant N$), and we prescribe $\varphi(x)$ to be of the form

$$\varphi(x) = \gamma x(x-\alpha)^2, \tag{7}$$

where $0 \leqslant \alpha \leqslant N$, and $\gamma > 0$ have to be chosen. Then clearly $\varphi(K) \geqslant 0$ for all $K \geqslant 0$, and by a result of Choi, Lam, and Reznick [3, Theorem 3.7], since $\deg P \leqslant 3$, this implies that $P \geqslant 0$ when $x_j \geqslant 0$.

We have

$$\varphi'(x) = \gamma(x-\alpha)(3x-\alpha),$$

and so, in order to impose $\varphi(x) \leqslant 1$, $0 \leqslant x \leqslant N$, all we have to require is $\varphi(\alpha/3) \leqslant 1$, $\varphi(N) \leqslant 1$, that is,

$$2\gamma\alpha^3/27 \leqslant 1$$
$$\gamma N(N-\alpha)^2 \leqslant 1.$$

We put $\alpha = \lambda N$ $(0 < \lambda < 1)$, and we obtain

$$\begin{aligned} 4\gamma\lambda^3 N^3/27 &\leqslant 1 \\ \gamma N^3(1-\lambda)^2 &\leqslant 1. \end{aligned} \tag{8}$$

But $\varphi(1) = \gamma(1-\alpha)^2 = \gamma(1-\lambda N)^2$ is an increasing function of γ. So $\varphi(1)$ will be maximal if both inequalities in (8) are equalities. Solving in λ, MAPLE gets

$$\left(\frac{\lambda}{3}\right)^3 = \left(\frac{1-\lambda}{2}\right)^2$$

and finds the solutions $\lambda = 3/4$, 3, 3.

This gives $\alpha = 3N/4$, $\gamma = 16/N^3$, and

$$\varphi(x) = \frac{16}{N^3}\, x\left(x - \frac{3}{4}N\right)^2.$$

The values of A, B, C, are now deduced from the system

$$\varphi(1) = A, \qquad \varphi(2) = 2A - 2B, \qquad C = 6\gamma,$$

easily solved by MAPLE.

We now show that $P \leqslant 1$ when all the x_j are $\leqslant 1$.

For this, we first observe that P can be written as

$$P(x_1, ..., x_N) = \frac{9}{N} \sum_1^N x_i^3 - \frac{24}{N^2} \left(\sum_1^N x_i^2\right)\left(\sum_1^N x_i\right) + \frac{16}{N^3}\left(\sum_1^N x_i\right)^3. \tag{9}$$

To simplify our notation, we put

$$m_1 = \frac{1}{N} \sum_1^N x_i, \qquad m_2 = \frac{1}{N} \sum_1^N x_i^2, \qquad m_3 = \frac{1}{N}\sum_1^N x_i^3,$$

and P becomes

$$P = 9m_1 - 24m_1 m_2 + 16m_3.$$

We have:

LEMMA 4. *Let P be a symmetric polynomial of degree 3, written as*

$$P = am_3 + bm_1 m_2 + cm_1^3,$$

where $a+b+c \leqslant 1$, $3a+b \geqslant 0$, $2b+3c \geqslant 0$. Then, if $P \geqslant 0$ when all $x_i \geqslant 0$, P automatically satisfies $P \leqslant 1$ when $0 \leqslant x_i \leqslant 1$.

Proof of Lemma 4. We set $x_i = 1 - t_i$. If $x_i \leqslant 1$, $t_i \geqslant 0$, and with the notation

$$\mu_1 = \frac{1}{N} \sum_1^N t_i, \qquad \mu_2 = \frac{1}{N} \sum_1^N t_i^2, \qquad \mu_3 = \frac{1}{N} \sum_1^N t_i^3,$$

P becomes

$$\begin{aligned} P = & -(a\mu_3 + b\mu_1 \mu_2 + c\mu_1^3) + a + b + c \\ & - 3\mu_1(a+b+c) + \mu_2(3a+b) + \mu_1^2(2b+3c). \end{aligned}$$

Since $a\mu_3 + b\mu_1\mu_2 + c\mu_1^3 \geqslant 0$, the condition $P \leqslant 1$ will be satisfied as soon as

$$a+b+c-3\mu_1(a+b+c)+\mu_2(3a+b)+\mu_1^2(2b+3c) \leqslant 1.$$

But $\mu_2 \leqslant \mu_1$, $3a+b \geqslant 0$, so this inequality holds if

$$a+b+c-\mu_1(1-\mu_1)(2b+3c) \leqslant 1,$$

which is satisfied by assumption. This proves Lemma 4, and finishes the proof of the theorem in the case $k=3$.

Remark. If we put

$$Q(x_1, ..., x_{2N}) = P(x_1, ..., x_N) - P(x_{N+1}, ..., x_{2N}),$$

we find that $|Q(x_1, ..., x_{2N})| \leqslant 1$ if $0 \leqslant x_j \leqslant 1$, and $|Q|_{\text{lead}} \to 18$ when $N \to +\infty$.

So the best constant C_3 in the inequality

$$|Q|_{\text{lead}} \leqslant C_k \sup_{0 \leqslant x_j \leqslant 1} |Q(x_1, ..., x_N)|$$

satisfies $C_3 \geqslant 18$; Theorem 1.2 in [1] shows that $C_k \leqslant 4k^2$.

Can this construction of P, with large leading coefficients, be carried over for $k>3$? We don't know. Following the same pattern would require:

— Finding a polynomial $\varphi(x)$ of degree k, in one variable, with $0 \leqslant \varphi(x) \leqslant 1$ if $0 \leqslant x \leqslant N$, and $\varphi(1)$ as large as possible;

— Identifying the many-variable polynomial $P(x_1, ..., x_N)$, homogeneous of degree k, with first term $A \sum_1^N x_i^k$, such that

$$P(\underbrace{1, ..., 1}_{j \text{ times}}, 0, ..., 0) = \varphi(j),$$

for $j = 0, ..., N$;

— Proving that $0 \leqslant P \leqslant 1$ if all x_j's satisfy $0 \leqslant x_j \leqslant 1$.

The first two steps are not very hard to perform, but the last one—proving that $0 \leqslant P \leqslant 1$— does not seem within our reach at present. Of course, the result of Choi, Lam, and Reznick we have used is not valid for $k \geqslant 3$, but this is not the main point: our proof, for $k = 3$, only uses this result for simplicity (our original proof did not). The main point is that no tool is presently known, ensuring that a many-variable polynomial, of degree $k > 3$, satisfies $0 \leqslant P \leqslant 1$ when all x_i satisfy $0 \leqslant x_i \leqslant 1$.

Since this problem cannot be solved, we have two possibilities. The first one is to build P, with $0 \leqslant P \leqslant 1$, by some iterative procedure from a known polynomial: this will lead to the estimate for B_k in Theorem 1. These estimates are not in k^2 as we would like, but they are better than anything previously known.

The second one will be to change the norm, and replace

$$\sup_{0 \leqslant x_j \leqslant 1} |P(x_1, ..., x_N)|$$

by the quantity

$$\sup_{x_j = 0, 1} |P(x_1, ..., x_N)|,$$

which will be discussed at the end of the paper.

We now turn to the iterative procedure in order to estimate B_k.

PROPOSITION 5. *Assume we can find a polynomial P_0, with N_0 variables, homogeneous of degree k_0, with the properties:*

— *all leading coefficients are* 1,

— *if all x_j satisfy $0 \leqslant x_j \leqslant 1$, then $0 \leqslant P_0 \leqslant 1$.*

Then $B_k \geqslant k^{\log N_0 / \log k_0}$, for all k of the form $k = k_0^j$, $j \in \mathbb{N}$.

Proof of Proposition 5. By assumption $|P_0|_{\text{lead}} = N$ and $0 \leqslant P_0 \leqslant 1$ if all $x_j \in [0, 1]$. Set $P_1 = P_0$, and

$$P_2 = P_0(P_1(x_1, ..., x_{N_0}), P_1(x_{N_0+1}, ..., x_{2N_0}), ..., P_1(x_{(N_0-1)N_0+1}, ..., x_{N_0^2})).$$

So P_2 has N_0^2 variables, $|P_2|_{\text{lead}} = N^2$, $\deg P_2 = k_0^2$, and $0 \leqslant P_2 \leqslant 1$ if $x_i \in [0, 1]$.

Assume P_{j-1} has been defined, with N_0^{j-1} variables, $\deg P_{j-1} = k^{j-1}$, and $0 \leqslant P_{j-1} \leqslant 1$ if $x_i \in [0, 1]$. Set

$$P_j = P_0(P_{j-1}(x_1, ..., x_{N_0^{j-1}}), ..., P_{j-1}(x_{(N_0-1)N_0^{j-1}+1}, ..., x_{N_0^j})).$$

So P_j has N_0^j variables, $|P_j|_{\text{lead}} = N_0^j$, $\deg P_j = k_0^j$, and $0 \leqslant P_j \leqslant 1$ if $x_i \in [0, 1]$.

Set $k = \deg P_j$. Then

$$N_0^j \leqslant B_k.$$

But $k = k_0^j$, $j = \log k/\log k_0$, and

$$B_k \geqslant N_0^{\log k/\log k_0} = k^{\log N_0/\log k_0},$$

as we announced. This proves Proposition 5.

We observe that, in order to be applied, this inductive procedure requires a polynomial with leading coefficients all equal to 1, and this is not the case of the ones we have exhibited so far.

So we will prove:

PROPOSITION 6. *The polynomial in 6 variables, with 56 terms,*

$$P_0(x_1, ..., x_6) = \sum_1^6 x_i^3 - \frac{1}{2} \sum_{i \neq j} x_i^2 x_j + \frac{1}{2} \sum_{i<j<k} x_i x_j x_k \tag{10}$$

satisfies $0 \leqslant P_0 \leqslant 1$ *if* $x_i \in [0, 1]$.

This proposition, producing a polynomial of degree 3 with 6 variables, gives the estimate $k^{\log 6/\log 3}$ in Theorem 1. It improves upon the estimate $k^{\log 3/\log 2}$, obtained by A. Tonge from the consideration of the polynomial

$$P_0(x, y, z) = x^2 + y^2 + z^2 - xy - yz - zx,$$

which also satisfies $0 \leqslant P_0 \leqslant 1$ if $x, y, z \in [0, 1]$.

Before proving Proposition 6, we will state:

PROPOSITION 7. *The polynomial* P_0 *defined in* (10) *is, among all polynomials of degree* 3 *with leading coefficients* 1, *with* 6 *variables, the only one* *which may satisfy* $0 \leqslant P_0 \leqslant 1$ *if all* $x_i \in [0, 1]$. *There is no such polynomial with 7 variables.*

Proof of Proposition 7. We consider any degree 3 polynomial with N variables, of the form

$$P = \sum_1^N x_i^3 - C_1 \sum_{i<j} x_i^2 x_j + C_2 \sum_{i<j<k} x_i x_j x_k.$$

Taking K of the variables x_i to be 1, $N-K$ to be 0, we obtain the set of conditions

$$0 \leqslant K - K(K-1)\, C_1 + \frac{K(K-1)(K-2)}{6}\, C_2 \leqslant 1,$$

which can be written, for $K \geqslant 2$,

$$\frac{1}{K} \leqslant C_1 - \frac{K-2}{6}\, C_2 \leqslant \frac{1}{K-1}. \tag{11}$$

Taking successively $K = 2, 3, 4$, we get

$$\tfrac{1}{2} \leqslant C_1 \leqslant 1,$$

$$C_2 \geqslant 6(C_1 - \tfrac{1}{2}) \geqslant 0,$$

$$C_2 \geqslant 1/2.$$

The left-hand side conditions in (11) can be written

$$C_1 \geqslant \frac{K-2}{6}\, C_2 + \frac{1}{K}, \tag{12}$$

and since $C_2 \geqslant 1/2$, the strongest one will be the one with the highest K.

The right-hand side gives

$$C_1 \leqslant \frac{K-2}{6}\, C_2 + \frac{1}{K-1}, \tag{13}$$

and the conditions for $K \geqslant 4$ are weaker than those for $K = 4$, and so we keep these for $K = 3, 4$, that is,

$$C_1 \leqslant \tfrac{1}{6} C_2 + \tfrac{1}{2} \tag{14}$$

$$C_1 \leqslant \tfrac{1}{3} C_2 + \tfrac{1}{3}. \tag{15}$$

This implies that no 7-variable polynomial may exist. Indeed, we would have by (12)

$$C_1 \geqslant \tfrac{5}{6} C_2 + \tfrac{1}{7},$$

and by (15)

$$\tfrac{5}{6} C_2 + \tfrac{1}{7} \leqslant \tfrac{1}{3} C_2 + \tfrac{1}{3},$$

which gives $C_2 \leqslant 8/21$, contradicting $C_2 \geqslant 1/2$.

This also implies the uniqueness for $K = 6$. Indeed, (12) gives

$$C_1 \geqslant \tfrac{2}{3} C_2 + \tfrac{1}{6},$$

and compatibility with (14), (15) implies $C_2 = 1/2$.

Coming back to (11), we find

$$\frac{1}{K} \leqslant C_1 - \frac{K-2}{12} \leqslant \frac{1}{K-1},$$

and for $K = 4$, this gives $C_1 \leqslant 1/2$, and finally $C_1 = 1/2$, which proves Proposition 7.

We now prove Proposition 6.

(1) To show that $P_0 \geqslant 0$ if $x_j \geqslant 0$, by the theorem of Choi, Lam, and Reznick [3] already cited, it is enough to do it when K of the variables are equal to 1, $6-K$ equal to 0 ($K = 0, ..., 6$). Set

$$\varphi(K) = P(\underbrace{1, ..., 1}_{K \text{ times}}, \underbrace{0, ..., 0}_{6-K \text{ times}}).$$

Then

$$\varphi(K) = K - \frac{K(K-1)}{2} + \frac{K(K-1)(K-2)}{12}$$

$$= \frac{1}{12} K(K-4)(K-5),$$

and $\varphi(K) \geqslant 0$ for $K = 0, ..., 6$.

(2) To show that $P_0 \leqslant 1$ if $x_j \in [0, 1]$, we write P_0 under symmetric form

$$P_0 = 10\left(\frac{1}{6} \sum_1^6 x_i^3\right) - 27\left(\frac{1}{6} \sum_1^6 x_i^2\right)\left(\frac{1}{6} \sum_1^6 x_i\right) + 18\left(\frac{1}{6} \sum_1^6 x_i\right)^3$$

$$= 10 m_3 - 27 m_1 m_2 + 18 m_1^3,$$

and we apply Lemma 4 again.

This concludes the proof of Proposition 7, and that of Theorem 1.

Remark. In a preliminary version of this paper, the proof of Proposition 5 was obtained by symbolic manipulation in the following way. Maple computes the 6 partial derivatives (which have degree 2), and the differences $\partial P/\partial x_i - \partial P/\partial x_j$. The entire system of differences is then solved. Then one studies the boundary cases $x_j = 0, 1$. The proof presented here is of course much simpler, but there is no evidence it exists for degree 5 and above.

We now investigate similar concepts for the quantity

$$\{P\}_{0,1} = \max_{x_j = 0, 1} |P(x_1, ..., x_N)|$$

and define D_k as the smallest constant such that

$$|P|_{\text{lead}} \leqslant D_k \max_{x_j = 0, 1} |P(x_1, ..., x_N)|$$

holds for all polynomials P, homogeneous of degree k, in many variables $x_1, ..., x_N$.

First, the proof of Theorem 1.2 in [1] shows that

$$D_k \leqslant 4k^2.$$

We are going to prove:

PROPOSITION 8. *For every* $k \geqslant 1$, $D_k \geqslant 2k^2$.

Proof of Proposition 8. We consider P under the form

$$\begin{aligned} P &= A_0\left(\frac{1}{N}\sum_1^N x_i^k\right) + A_1\left(\frac{1}{N}\sum_1^N x_i^{k-1}\right)\left(\frac{1}{N}\sum_1^N x_i\right) \\ &\quad + A_2\left(\frac{1}{N}\sum_1^N x_i^{k-2}\right)\left(\frac{1}{N}\sum_1^N x_i\right)^2 \\ &\quad + \cdots + A_j\left(\frac{1}{N}\sum_1^N x_i^{k-j}\right)\left(\frac{1}{N}\sum_1^N x_i\right)^j + \cdots + A_{k-1}\left(\frac{1}{N}\sum_1^N x_i\right)^k \\ &= A_0 m_k + A_1 m_{k-1} m_1 + \cdots + A_j m_{k-j} m_1^j + \cdots + A_{k-1} m_1^k, \end{aligned}$$

with our previous notation.

The coefficient of x_1^k is

$$\frac{A_0}{N} + \frac{A_1}{N^2} + \cdots + \frac{A_{k-1}}{N^k},$$

and therefore

$$|P|_{\text{lead}} = A_0 + \frac{A_1}{N} + \cdots + \frac{A_{k-1}}{N^{k-1}} \to A_0,$$

when $N \to +\infty$.

If M of the variables take the value 1, and the other $N-M$ the value 0, we have

$$P(\underbrace{1, ..., 1}_{M \text{ times}}, \underbrace{0, ..., 0}_{N-M \text{ times}}) = A_0 \frac{M}{N} + A_1 \left(\frac{M}{N}\right)^2 + \cdots + A_{k-1} \left(\frac{M}{N}\right)^k,$$

and so, if we set

$$f(x) = A_0 x + A_1 x^2 + \cdots + A_{k-1} x^k,$$

we want $0 \leqslant f(x) \leqslant 1$ if $0 \leqslant x \leqslant 1$, and A_0 maximal.

But $A_0 = f'(0)$, and the solution of this problem is given by the Chebyshev polynomial T_k (see Rivlin [4]).

So we take $f(x) = ((-1)^{k-1} T_k(2x-1)+1)/2$, and since $-1 \leqslant T_k \leqslant 1$, we have $0 \leqslant f \leqslant 1$ on $[0, 1]$. Also, $f(0) = 0$, and $f'(0) = T'_k(-1) = k^2$ (see Rivlin [4, p. 105]).

Finally, we set

$$Q(x_1, ..., x_{2N}) = P(x_1, ..., x_N) - P(x_{N+1}, ..., x_{2N}),$$

and we obtain the announced estimate.

We observe that the coefficients $A_0, ..., A_{k-1}$ can be explicitly computed from the coefficients of the Chebyshev polynomial. In fact, P can be written as

$$P(x_1, ..., x_N) = \frac{1}{N} \sum_{i=1}^{N} \frac{x_i^{k+1}}{m_1} f(m_1/x_i).$$

The quantity $\{P\}_{0,1}$ is of course much easier to compute than any of the existing norms; however, it is not a norm: $\{x^2y - xy^2\}_{0,1} = 0$.

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